A Semi – Intuitive description of Sampling Rate Conversion and Multi-rate Processing in Digital Signal Processing

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1.0 Introduction: Multi-rate processing, sampling rate conversion, or *interpolation and decimation* as it is known, is a clever technique in DSP. As analog and mixed signal design engineers we have learned to use this technique in various product designs for our customers. It offers an added degree of freedom in the design of mixed signal integrated circuits that may be of help to other professionals such as ourselves.

Multi-rate processing finds use in signal processing systems where various sub-systems with differing sample or clock rates need to be interfaced together. At other times multi-rate processing is used to reduce computational overhead of a system. For example, an algorithm requires k operations to be completed per cycle. By reducing the sample rate of a signal or system by a factor of M, the arithmetic bandwidth requirements are reduced from kf_s operations to kf_s/M operations per second.

In other applications, re-sampling a signal at a lower rate will allow it to pass through a channel of limited bandwidth. In another application a high accuracy Delta – Sigma A/D converter can be made with a very high modulation rate at the front end followed by a decimator (down converter) to reduce the sampling rate and provide converted samples at or near the Nyquist rate.

Applications for this technique abound, *if it is understood by the practitioner*. The challenge is that it is not easy to pick up a book or a paper on DSP and understand Decimation and Interpolation to an intuitive extent. This causes hesitation in usage.

We have found the techniques to be useful in mixed signal designs. The article attempts to explain the technique. It may be viewed as an augment to existing books and papers written by our learned colleagues. The hope is that once understood, it may lead to a number of exciting new applications in practical design. At least this is the authors' earnest hope. Perhaps we can look forward to hearing from others who may show us how to use multi-rate sampling in a variety of new ways.

2.0 Types of sampling rate conversion: There are three types of sampling rate conversion. These are:

- *Down Conversion or Decimation by a factor M*
- *Up Conversion or Interpolation by a factor L*
- *Sampling Rate conversion by a ratio of M and L*

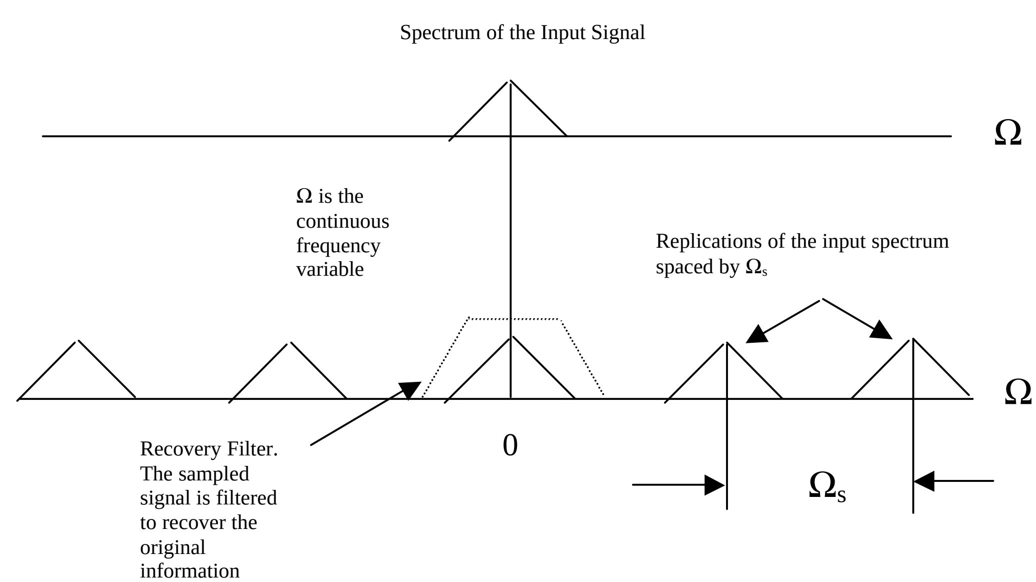
3.0 Down Conversion or Decimation: We have found that it is easier to understand what is going on in sampling rate conversion by starting with what the spectrum of a sampled signal looks like. (Please refer to the appendices for supporting information about sampling, discrete variables etc.)

The spectrum of a sampled signal can be written:

$$X_s(j\Omega) = (1/T)\Sigma\ X_c(j\Omega - jk\Omega_s) \qquad (1)$$

The summation is over k from $-\infty$ to ∞, T = time period of the sampling frequency Ω_s

What happens is shown in the figure below:

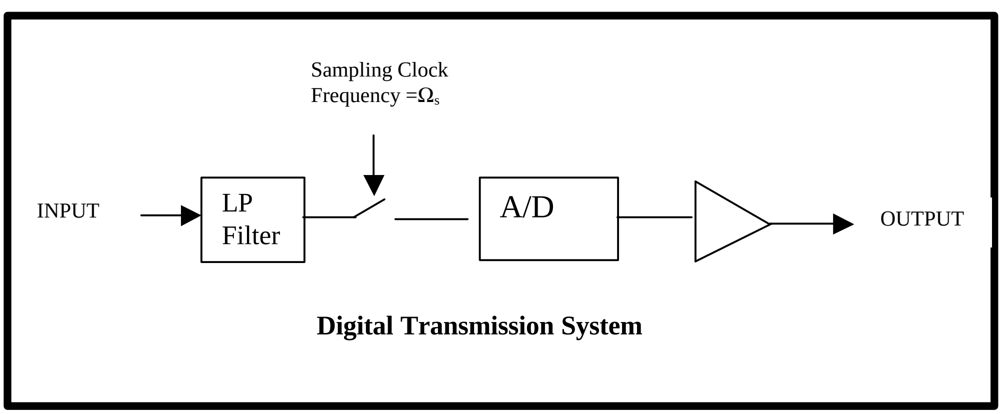


Effect of Periodic Sampling

The sampling process simply generates replicas of the original signal everywhere along the frequency axis. The separation between the replicas (also known as *aliases*) is the same as the sampling frequency.

From the picture above it is fairly easy to see that if the sampling frequency is not large enough the replicas will start *overlapping.* As the sampling frequency increases the replicas start moving away from each other. At the points of overlap the information will be *corrupted.* This must be avoided in sampling. The rule is, *keep the sampling frequency at least 2X the maximum bandwidth of the signal you want to sample. (Or limit the bandwidth by pre-filtering)* In practice, of course, this sampling frequency may be 4X or 8X or even higher. In Delta – Sigma or over-sampled systems it may be as high as 256X (which confers some benefits on the design sometimes). The replicas contain all the information that is present in the original signal, except for a scaling of the amplitude of the signal.

The question is, what does all this mean in practical terms? In order to get an intuitive understanding of decimation lets assume that a data signal is being transmitted down a cable. This data is generated by sampling an analog signal, and following the sampler with an A/D converter and a buffer as shown below:



Note, that the sampler is simply a switch controlled by the sampling clock. A pre-filter has been inserted between the input and the sampler

to limit the bandwidth of the input and to get rid of accompanying noise which can cause problems with the sampled signal.
The digital signal (numbers) is then transmitted down the physical link. The buffer simply overcomes the attenuation of the physical media. Note that in the simplest case the digital signal is being sent at the sampling rate. If the receiver can operate at the same sampling rate (or higher), then all that has to be done is to read the received digital signal into the receiver and extract the original signal simply by low pass filtering the signal.

However, if the receiver operates at a much slower rate, say 3X slower then it can only read one out of 3 samples of the signal. Can this signal still be received properly?

One way to solve this problem to first use a D/A converter, convert to an analog signal and then re-sample the reconstructed analog signal at a slower rate (3X slower) . This means we need more power, more circuitry, less reliability etc.

Another way is to simply use digital re-sampling. In this case using decimation. The following describes what is done and its effects and ramifications.

Fundamentally assume a 3X decimation. In other words only one out of 3 samples are chosen and the rest are discarded. Therefore there are actually two signals that have to be considered. One is the old signal running at the old sampling rate, and the other is the new signal running at the new sampling rate. Let these variables be: $x(n)$, $x_d(n)$, T and T_{old} .

Let $3T_{old} = T$

The factor 3, is the decimation factor, typically labeled “M”.
We know from the description of the sampling process given above, what old signal looks like in the frequency domain. The question is what does the *new signal* look like in the same domain. (It is actually easier to work in the frequency domain than the time domain in this case). To see this we have to go through some mathematical manipulations first and then describe the effects pictorially and see what must be done to decimate at 3X.

Starting from the frequency domain expression for the sampled signal which is repeated here for convenience.

$X_s(j\Omega) = (1/T)\Sigma\, X_c(j\Omega - jk\Omega_s)$ (2)

The summation is over k from -∞ to ∞, T = time period of the sampling frequency Ω_s

Let us change the continuous frequency variable to the discrete frequency variable we get: (See the appendix or insert for continuous and discrete frequency variables and mappings)

$X(e^{j\omega})$ = Spectrum of x(n)

= $1/T\Sigma\, X_c(j\omega/T_{old} - jk2\pi/T_{old})$ (3)

The summation is over k from -∞ to ∞, T = time period of the sampling frequency Ω_s

Now a down sampler will have a sampling frequency of $1/MT_{old}$. In our case M=3. So our down sampler has a frequency of:

$1/3T_{old}$, T_{old} = old sampling frequency.

So now we re- sample the input with this new sampling frequency. The new spectrum is then:

$1/3T_{old}\Sigma\, X_c(j\omega/3T_{old} - jr2\pi/3T_{old})$

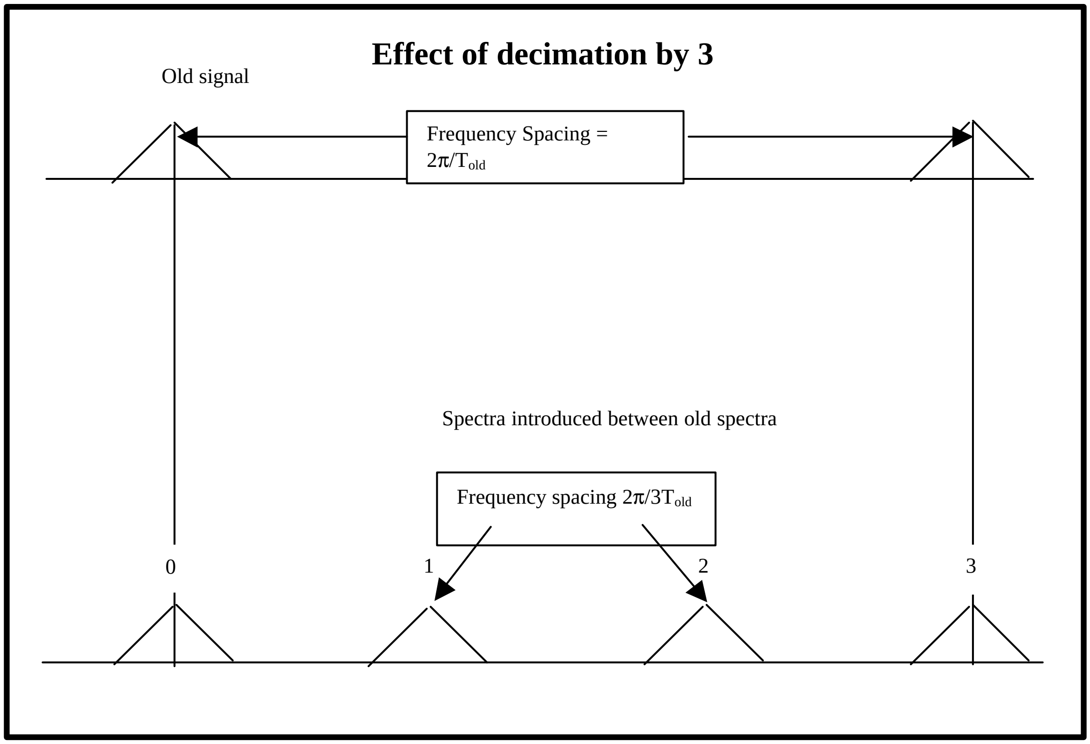
The summation is over r from -∞ to ∞, T = time period of the sampling frequency Ω_s

Notice that the summation variable has changed from k to r. r is the new summation variable and is closely related to k via the following transformation.

r = i + kM

Using these three expressions the frequency domain behavior of decimation can be explored.

First, the frequency spacing between replications of the decimated signal is one third of the old signal from the equation of the frequency spectra of the decimated signal. Pictorially the situation is depicted in the figure below:

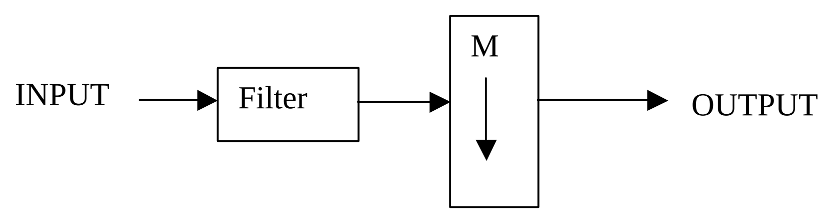


As can be seen from the figure above the sampling rate can be decreased and again a low pass filter can be used to extract the information. The effect is the introduction of spectra in between the replications of the old spectra.

The relationship above defines how many spectra are introduced. The counter i goes from 0 to 2 (M=3). Since M = 3 we have the spectra number r being defined by k and M.

Refer to the figure above. Let k =0. Then r = i. The decimation factor for the example above is M=3. Then the number of spectra that fit in one old sampling frequency space ($2\pi/T_{old}$ space) can be seen to be 3. Specifically, k=0, r=0,1,2. The k=1, r=3,4,5, then k=2 and r=6,7,8 and so on. i determines the number of spectra between each *old* spectra spacing.

We must be aware , that the rules for Nyquist rate sampling still apply. There is a limit to the decimation factor. We can only decimate till aliasing starts. So knowing the old spacing, and the bandwidth of the signal we can choose a decimation factor M which will allow us to reduce the sampling rate and yet avoid aliasing and corruption of the re-sampled signal. Or alternatively we can limit the bandwidth of the signal before we resample to achieve the same effect. Therefore the generic system diagram of a down sampler is that shown below.



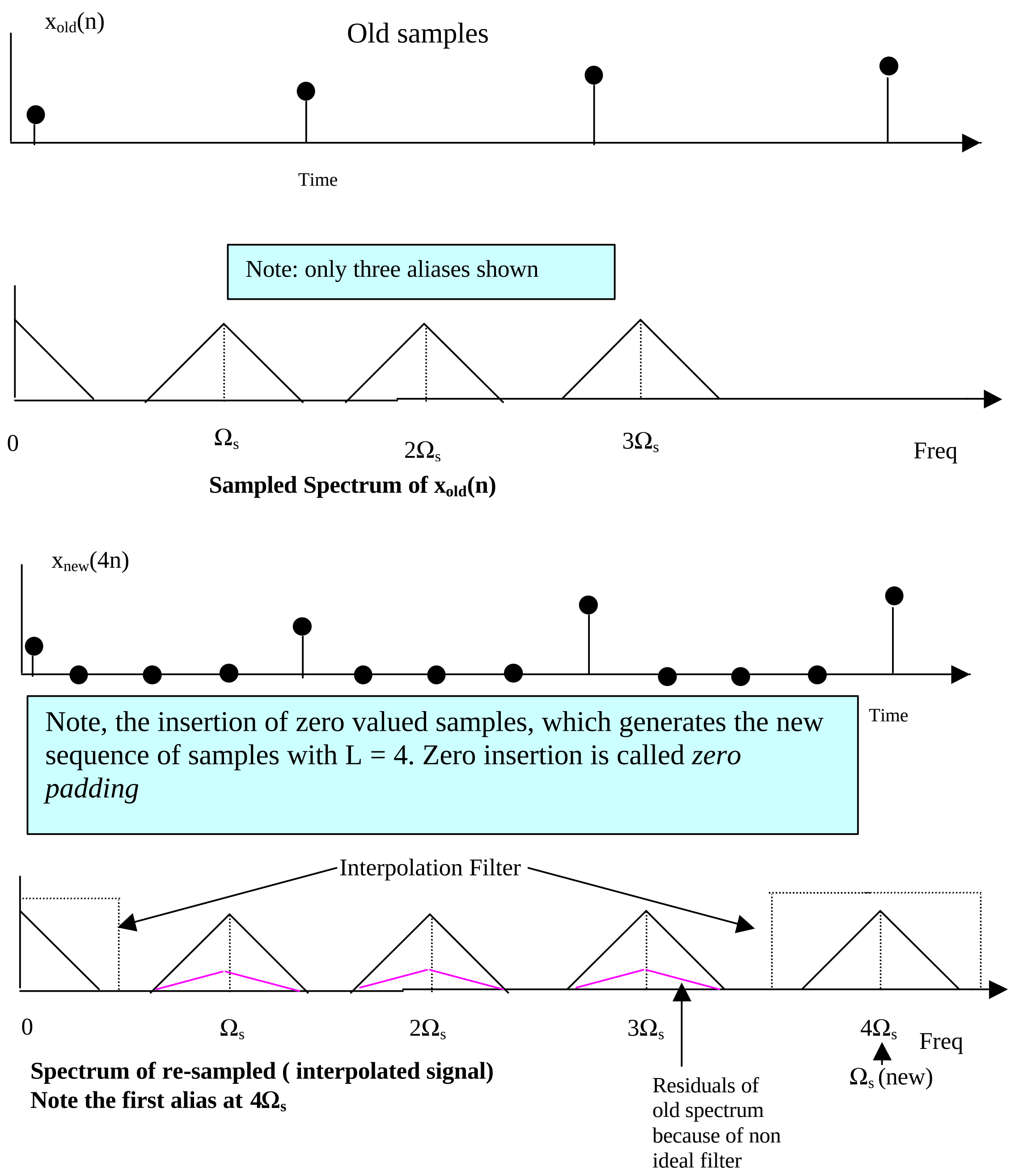
Decimator system

Note that some practical considerations have been left out in this description of decimation. However, the idea here is to present the techniques in a semi – intuitive fashion so that the main idea can be grasped easily and " gut feel" can be developed. Once this happens the more practical aspects/concerns of the technique can be not only understood, but also taken care of as they arise.

4.0 Interpolation or Up Conversion: Let us now consider sampling rate up conversion or interpolation. In interpolation the sampling rate is actually increased. This is a slightly more intricate operation than decimation since we need more samples than that originally available. The process is illustrated in the following diagrams with accompanying descriptions so that it can be understood intuitively.

In these diagrams and descriptions the sampling rate increase will be by a factor of 4. The up sampling factor is L. In this example L=4. Again a frequency domain description is well suited to understanding the operation of up sampling.

Pictorially interpolation is represented by the figures below.



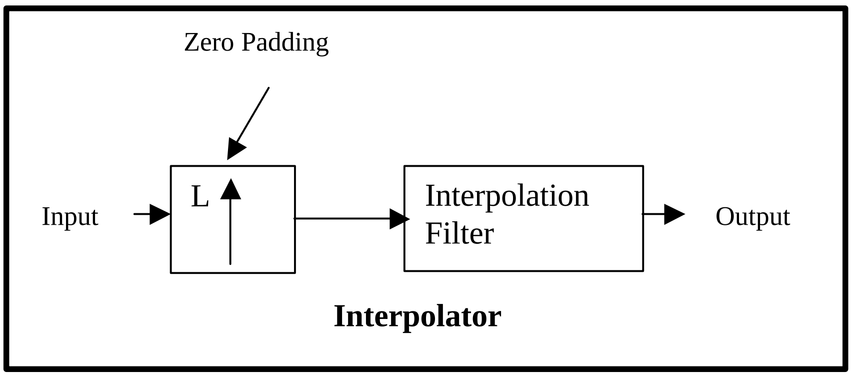
Interpolation at L = 4

With reference to the figure above, the process of interpolation is carried out as follows:

1.0 Insert the requisite number of zero's between each two samples of the old sampled signal. (3 new zero valued samples for L = 4).
2.0 Establish the new index for the interpolated sequence. This is described below.
3.0 Filter out the old spectrum by using a filter centered at 0 Hz and Ω_s(new). This is called the interpolation filter. The output of the filter is the desired up sampled digital signal or sequence.

The output of the interpolation filter will contain residuals of the old spectrum as shown in the figure, since the filter cannot be ideal. Thus the performance of the interpolation depends critically on the interpolation filter.

The interpolator block diagram is thus as shown below.



The mathematics of interpolation is analogous to that of decimation and is shown below.

The old signal is given by:

$x_{old}(n)$ where n is the sample index

The up sampled signal is given by

$$x_{new}(n) = x_{old}(n/L) \text{ if } n = +/-0, +/-L, +/-2L \text{ etc}$$
$$= 0 \text{ otherwise.}$$

In the frequency domain, we get for the up sampled signal (from the Fourier Response)

$$X_{new}(e^{j\omega}) = \sum x_{old}(n)e^{-jnL\omega}$$

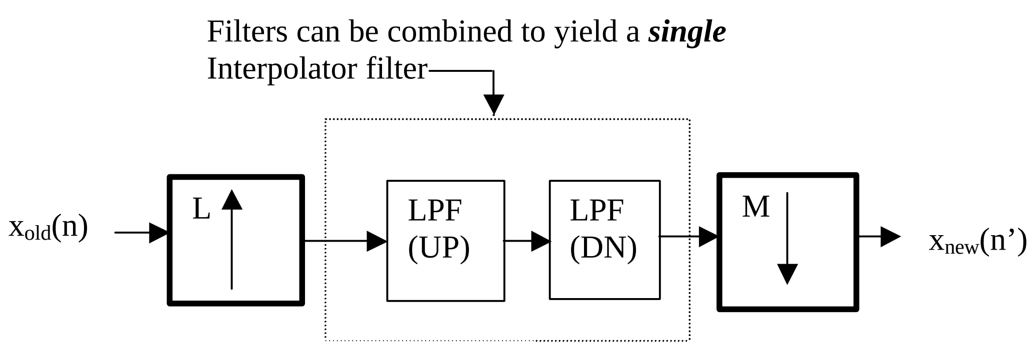
Summation over n

This shows that the old spectrum is simply scaled in frequency as shown in the diagram.

A time domain explanation may also be useful. In the time domain the low pass filter at the output simply smoothes out the waveform or interpolates between samples at integer multiples of L.

There are many applications for interpolation. Some of the obvious ones are in D/A conversion and recording from a CD to a Digital Audio Tape. The CD provides data at 44.1 kHz while the DAT needs data at 48 kHz. This is a perfect application for up sampling. In general any interfacing of digital signals from a lower frequency source to a higher frequency sink can and should use interpolation in the digital domain. This can save power and silicon space thereby rendering the product more efficient and cost effective.

4.0 Sampling rate conversion by a ratio of M and L: The sampling rate conversion by ratios of L and M is a solution to the problem of changing sampling rates by a factor that is not an integer. Since we are free to choose L and M, we can change sample rates by almost any factor in practice. Simply choose L and M appropriately. This technique is shown pictorially in the figure below.



Fractional re-sampling (L/M)
Interpolate and then decimate

Note that it is entirely possible to combine the two filters together and thus have a single filter between the interpolator and the decimator. The combination interpolator and decimator is also called a *sample rate converter*. In other words L > D gives us an interpolator and when D>L we have a decimator. The purpose of the filter has been explained above.